Triplet order parameter of the triangular Ising model

1975 J. Phys. A: Math. Gen. 8 1797

(http://iopscience.iop.org/0305-4470/8/11/015)

J. Phys. A: Math. Gen., Vol. 8, No. 11, 1975. Printed in Great Britain. © 1975

Triplet order parameter of the triangular Ising model

Rodney Baxter†

Department of Physics, University of Edinburgh, James Clerk Maxwell Building, The King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, UK

Received 30 April 1975

Abstract. We consider the usual Ising model with pair interactions between nearest-neighbour spins σ on the triangular lattice. We evaluate the correlation $M_3 = \langle\sigma_1\sigma_2\sigma_3\rangle$, where σ_1, σ_2, σ_3 are the three spins at the vertices of a triangular face. This can be thought of as an order parameter associated with a three-spin 'field'. It vanishes at and above T_c, and has a critical exponent of $\frac{1}{8}$.

1. Introduction

There has recently been an interest in statistical mechanical lattice models with multi-spin interactions. This is due in part to the unusual behaviour of the eight-vertex model (Baxter 1972), which has continuously varying critical exponents and can be thought of as an Ising model with two- and four-spin interactions (Wu 1971, Kadanoff and Wegner 1971).

Wood and Griffiths (1972) considered an Ising model on the triangular lattice, with both pair interactions between nearest-neighbour sites i and j, and triplet interactions between spins at the vertices i, j, k of a triangular face. Including also a magnetic field H, the Hamiltonian of this model is

$$E = H\sum\sigma_i - J\sum\sigma_i\sigma_j - J'\sum\sigma_i\sigma_j\sigma_k, \tag{1}$$

where the first summation is over all N sites of the lattice (ie $i = 1, \ldots, N$), the second over all $3N$ edges, and the third over all $2N$ triangular faces.

The partition function Z and free energy per site f are then given by

$$Z = \sum_{\{\sigma\}} \exp(-\beta E), \tag{2}$$

$$f = -kT \lim_{N\to\infty} N^{-1} \ln Z, \tag{3}$$

where $\beta = 1/kT$ is the Boltzmann factor, T is the temperature, and the summation in (2) is over all values ± 1 of the N spins σ_i.

The free energy of the model can be evaluated exactly when any two of H, J, J' are zero. In particular, when $H = J = 0$ we have the pure three-spin model, the free energy of which was obtained by Baxter and Wu (1973, 1974). In this case H and J can be regarded as 'fields' and very plausible conjectures have recently been obtained for the

† Permanent address: Research School of Physical Sciences, The Australian National University, Canberra, Australia.

corresponding zero-field spontaneous order parameters, using series expansions (Baxter *et al* 1975).

The other interesting limiting case, the one considered here, is when $H = J' = 0$. We then regain the normal two-spin triangular Ising model, for which the free energy, spontaneous magnetization, and nearest-neighbour correlation are of course known (Houtappel 1950, Husimi and Syozi 1950, Wannier 1950, Potts 1955, Green 1963, Stephenson 1964).

However, this approach introduces a new order parameter for the two-spin model. We can now regard J_3 as a field and define an associated 'three-spin spontaneous magnetization' M_3 by

$$M_3 = -\lim_{J'\to 0}\left(\frac{\partial f}{\partial J'}\right)_{H=0}, \tag{4}$$

where the limit is taken through positive values of J'. From the above it is readily seen that

$$M_3 = \langle \sigma_i \sigma_j \sigma_k \rangle,$$

where i, j, k are the three vertices of a triangular face.

From the spin-reversal symmetry we expect M_3 to be zero at temperatures above the critical temperature T_c, while from series expansions we expect it to be positive below T_c. Thus it behaves similarly to the normal magnetization M associated with the field H.

It is therefore interesting to calculate M_3, and we do this in the following sections. In these we allow J to have different values in each of the three directions of edges of the lattice. The full result is given in equation (62), but here we remark that for the isotropic case we obtain

$$M_3 = M\left[3\frac{1+u}{1-u} - 2\left(\frac{1+3u}{(1+u)^3}\right)^{1/2}\right], \tag{5}$$

where

$$u = \exp(-4\beta J),$$

and M is the usual single-spin spontaneous magnetization, given by

$$M = \left(\frac{(1+u)^3(1-3u)}{(1-u)^3(1+3u)}\right)^{1/8}. \tag{6}$$

As T increases from 0 to T_c, u increases from 0 to $\frac{1}{3}$. Thus at T_c

$$M_3/M = 0{\cdot}803847\ldots, \tag{7}$$

so M_3 has the same critical exponent as M, ie $\frac{1}{8}$.

The result (5) provides a useful check on the series expansion for the general problem, where H, J, J' are all nonzero (Sykes and Watts 1975). This problem has of course *not* been solved exactly.

2. Determinant expression for M_3

We use the notation of Stephenson (1964, to be referred to as S). Draw the triangular lattice as in figure 1 and let $-J_1$, $-J_2$, $-J_3$ be the three nearest-neighbour interaction

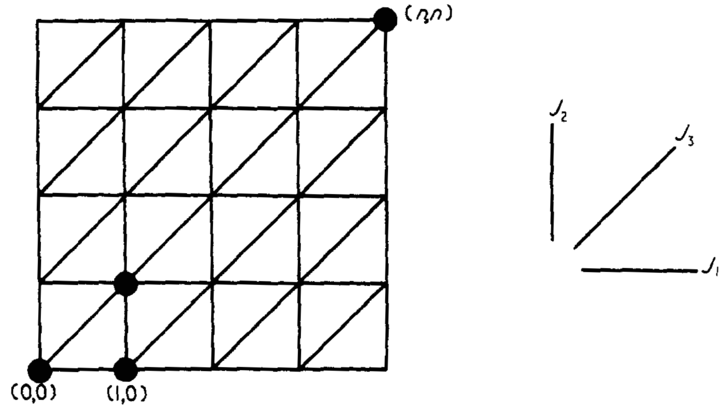

Figure 1. The triangular lattice, showing the labelling of sites and the directions in which the interactions J_1, J_2, J_3 apply.

energies, as indicated. Define $\mathscr{R}_n$ to be the ratio of two correlations as follows:

$$\mathscr{R}_n = \langle \sigma_{0,0}\sigma_{1,0}\sigma_{1,1}\sigma_{n,n}\rangle / \langle \sigma_{1,1}\sigma_{n,n}\rangle. \tag{8}$$

In the limit of n large we expect these correlations to factor, giving

$$\lim_{n\to\infty} \mathscr{R}_n = M_3/M, \tag{9}$$

where

$$M = \langle \sigma_{1,1}\rangle, \qquad M_3 = \langle \sigma_{0,0}\sigma_{1,0}\sigma_{1,1}\rangle. \tag{10}$$

Thus M_3 is our required order parameter and M is the usual spontaneous magnetization, given by equation (7.13) of S.

We can evaluate the correlations in (8) by Pfaffians, using the techniques of § 3 of S. To evaluate the four-spin correlation we perturb the horizontal dimer between sites (0, 0) and (1, 0), and all the diagonal dimers between (1, 1) and (n, n). This gives

$$\langle \sigma_{0,0}\sigma_{1,0}\sigma_{1,1}\sigma_{n,n}\rangle = v_1 v_3^{n-1}(|y^{-1}+Q|\,|y|)^{1/2}, \tag{11}$$

where

$$v_i = \tanh(J_i/kT), \qquad i = 1, 2, 3, \tag{12}$$

and if $t_i = v_i^{-1} - v_i$, then

$$y = \begin{pmatrix} 0 & t_1 & \mathbf{0} & \mathbf{0} \\ -t_1 & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -t_3 \boldsymbol{I} \\ \mathbf{0} & \mathbf{0} & t_3 \boldsymbol{I} & \mathbf{0} \end{pmatrix} \tag{13}$$

$$\boldsymbol{Q} = \begin{pmatrix} 0 & p & \boldsymbol{x}_1^{\mathrm{T}} & \boldsymbol{x}_3^{\mathrm{T}} \\ -p & 0 & \boldsymbol{x}_2^{\mathrm{T}} & \boldsymbol{x}_4^{\mathrm{T}} \\ -\boldsymbol{x}_1 & -\boldsymbol{x}_2 & \mathbf{0} & \boldsymbol{S} \\ -\boldsymbol{x}_3 & -\boldsymbol{x}_4 & -\boldsymbol{S}^{\mathrm{T}} & \mathbf{0} \end{pmatrix}. \tag{14}$$

Both matrices $\boldsymbol{y}$ and $\boldsymbol{Q}$ are $2n \times 2n$, $\boldsymbol{I}$ in (13) is the unit $n-1 \times n-1$ matrix, while in (14) p is the scalar, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_4$ are $(n-1)$-dimensional vectors, and $\boldsymbol{S}$ is a square $n-1 \times n-1$ matrix. Their elements are

$$\begin{aligned} p &= [1,0]_{rl} \\ (\boldsymbol{x}_1)_\alpha &= [\alpha,\alpha]_{rs} \\ (\boldsymbol{x}_2)_\alpha &= [\alpha-1,\alpha]_{ls} \\ (\boldsymbol{x}_3)_\alpha &= [\alpha+1,\alpha+1]_{rt} \\ (\boldsymbol{x}_4)_\alpha &= [\alpha,\alpha+1]_{lt} \\ S_{\alpha,\beta} &= [\beta-\alpha+1,\beta-\alpha+1]_{st}, \end{aligned} \tag{15}$$

where $\alpha, \beta = 1, \ldots, n-1$ and the right-hand side has the same meaning as in equation (4.2) of S.

We have used the antisymmetry of $\boldsymbol{Q}$ and the vanishing of $[\alpha,\alpha]_{ss}$ and $[\alpha,\alpha]_{tt}$ (equation (4.4) of S).

The two-spin correlation $\langle\sigma_{1,1}\sigma_{n,n}\rangle$ is also given by the right-hand side of (11), but with v_1 and the first two rows and columns of $\boldsymbol{y}$ and $\boldsymbol{Q}$ deleted.

It is trivial to evaluate $|\boldsymbol{y}|$ and $\boldsymbol{y}^{-1}$. We can simplify $|\boldsymbol{y}^{-1}+\boldsymbol{Q}|$ by using the general identity

$$\begin{vmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{D}|\,|\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}|, \qquad |\boldsymbol{D}| \neq 0. \tag{16}$$

This is true for any square matrices $\boldsymbol{A}$, $\boldsymbol{D}$. Here $\boldsymbol{A}$ is 2×2, $\boldsymbol{D}$ is $2n-2 \times 2n-2$. From (8), (11), (13), (14) we obtain

$$\mathscr{R}_n = v_1 t_1 (t_1^{-1} - p + \boldsymbol{x}_4^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{x}_1 - \boldsymbol{x}_3^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{x}_2) \tag{17}$$

where

$$\boldsymbol{B} = t_3^{-1}\boldsymbol{I} + \boldsymbol{S}. \tag{18}$$

From (6.4 and 9) of S, with $1+v_3^2$ corrected to $1-v_3^2$, it follows that

$$\boldsymbol{B} = (v_3 t_3)^{-1}\boldsymbol{A}, \tag{19}$$

where $\boldsymbol{A}$ is the $n-1 \times n-1$ Toeplitz matrix with elements

$$A_{\alpha,\beta} = a_{\beta-\alpha} = v_3\delta_{\alpha,\beta} + (1-v_3^2)[\beta-\alpha+1,\beta-\alpha+1]_{st}. \tag{20}$$

From equations (6.12) and (5.13) of S

$$a_\alpha = (2\pi)^{-1}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}\alpha\omega} A(\omega)\,\mathrm{d}\omega, \tag{21}$$

where

$$A(\omega) = \left(\frac{a - b\,\mathrm{e}^{\mathrm{i}\omega} - c\,\mathrm{e}^{-\mathrm{i}\omega}}{a - b\,\mathrm{e}^{-\mathrm{i}\omega} - c\,\mathrm{e}^{\mathrm{i}\omega}}\right)^{1/2}, \tag{22}$$

$$\begin{aligned} a &= 2v_3(1+v_1^2)(1+v_2^2) + 4v_1v_2(1+v_3^2), \\ b &= v_3^2 c = v_3^2(1-v_1^2)(1-v_2^2). \end{aligned} \tag{23}$$

Except at T_c, $A(\omega)$ has no branch-points on the real ω axis. It is to be chosen to be single-valued and continuous on the real ω axis, and to be positive when $\omega = \pi$.

It remains to give explicit expressions for the first five matrix elements on the right-hand side of (15). From (4.1), (4.2) and (2.12) of S

$$[\alpha, \beta]_{jk} = \frac{1}{4\pi^2}\iint_{-\pi}^{\pi} \exp[-\mathrm{i}(\alpha\phi_1+\beta\phi_2)]\frac{C_{kj}(\phi_1, \phi_2)}{\Delta(\phi_1, \phi_2)}\,\mathrm{d}\phi_1\,\mathrm{d}\phi_2 \tag{24}$$

where

$$\begin{aligned}\Delta(\phi_1, \phi_2) = {} & (1+v_1^2)(1+v_2^2)(1+v_3^2)+8v_1v_2v_3-2v_1(1-v_2^2)(1-v_3^2)\cos\phi_1 \\ & -2v_2(1-v_3^2)(1-v_1^2)\cos\phi_2-2v_3(1-v_1^2)(1-v_2^2)\cos(\phi_1+\phi_2)\end{aligned} \tag{25}$$

and from the appendix of S

$$\begin{aligned}C_{lr}(\phi_1, \phi_2) = {} & (1-v_2^2)(1-v_3^2)-4v_2v_3\,\mathrm{e}^{\mathrm{i}\phi_1}-v_1\,\mathrm{e}^{\mathrm{i}\phi_1}[(1+v_2^2)(1+v_3^2) \\ & +2v_3(1-v_2^2)\cos(\phi_1+\phi_2)+2v_2(1-v_3^2)\cos\phi_2],\end{aligned} \tag{26}$$

$$\begin{aligned}C_{sr}(\phi_1, \phi_2) = {} & -C_{tl}(-\phi_1, -\phi_2) = -1-v_2^2-2v_1v_2v_3 \\ & +[v_1v_3(1+v_2^2)+2v_2]\,\mathrm{e}^{-\mathrm{i}\phi_2}+v_3(1-v_2^2)\exp[-\mathrm{i}(\phi_1+\phi_2)] \\ & +v_1(1-v_2^2)\,\mathrm{e}^{\mathrm{i}\phi_1},\end{aligned} \tag{27}$$

$$\begin{aligned}C_{sl}(\phi_1, \phi_2) = {} & C_{tr}(-\phi_1, -\phi_2) = -(1-v_2^2)+v_1v_3(1-v_2^2)\exp[-\mathrm{i}(2\phi_1+\phi_2)] \\ & +(v_3+2v_1v_2+v_3v_2^2)\exp[-\mathrm{i}(\phi_1+\phi_2)]+(v_1+2v_2v_3+v_1v_2^2)\,\mathrm{e}^{-\mathrm{i}\phi_1}.\end{aligned} \tag{28}$$

Following Stephenson, equation (4.7), we make the substitutions $\phi_1 = \omega+\theta$, $\phi_2 = -\theta$. We then perform the θ integrations in each of the required integrals (24) by calculating the residues of poles inside the unit circle in the complex $\exp(\mathrm{i}\theta)$ plane. After some considerable algebra we find the results can be expressed as

$$p = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[-\frac{1}{v_1}+\frac{v_2}{v_1}D(-\omega)+(1-v_3^2)L^{\dagger}(\omega)R(\omega)\right]\mathrm{d}\omega, \tag{29}$$

$$(x_j)_\alpha = (2\pi)^{-1}\int_{-\pi}^{\pi}\mathrm{e}^{-\mathrm{i}\alpha\omega}I_j(\omega)\,\mathrm{d}\omega, \tag{30}$$

where

$$\begin{aligned} I_1(\omega) &= R(\omega)/A(\omega), & I_2(\omega) &= L(\omega)/A(\omega), \\ I_3(\omega) &= -R(\omega), & I_4(\omega) &= -L(\omega),\end{aligned} \tag{31}$$

and if

$$\begin{aligned} y_1 &= v_1(1+v_2^2)+v_2v_3(1+v_1^2), \\ y_2 &= v_2(1+v_1^2)+v_1v_3(1+v_2^2),\end{aligned} \tag{32}$$

then $D(\omega)$, $R(\omega)$, $L(\omega)$, $L^{\dagger}(\omega)$ are defined by

$$D(\omega) = [v_2(1-v_1^2)+v_1(1-v_2^2)\,\mathrm{e}^{\mathrm{i}\omega}]^{-1}, \tag{33}$$

$$R(\omega) = \tfrac{1}{2}[y_1D(\omega)+v_3-(y_2D(\omega)+1)A(\omega)]/(1-v_3^2), \tag{34}$$

$$L(\omega) = \tfrac{1}{2}[y_1D(\omega)-v_3-(y_2D(\omega)-1)A(\omega)]/v_1(1-v_3^2), \tag{35}$$

$$L^{\dagger}(\omega) = \tfrac{1}{2}[y_1D(-\omega)-v_3+(y_2D(-\omega)-1)A(-\omega)]/v_1(1-v_3^2). \tag{36}$$

Note that in the interesting case $v_1 = v_2$, $D(\omega)$ has a simple pole at $\omega = \pi$. Nevertheless all the integrands in (29) and (30) are analytic on and about the real axis in the ω plane. In our subsequent analysis we take care not to separate out terms which have this potential pole.

We now have explicit integral expressions for p and the elements of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_4$, $\boldsymbol{A}$. Thus we can in principle calculate $\mathscr{R}_n$ from (17) and (19). The chief difficulty is the inversion of the matrix $\boldsymbol{B}$, or $\boldsymbol{A}$. However, it turns out that we can handle the desired limit $n \to \infty$, as we show in the next section.

First we perform some algebraic manipulations to simplify the expression for $\mathscr{R}_n$. Substituting the expressions (30) and (19) into (17), we obtain

$$\mathscr{R}_n = v_1 - (1-v_1^2)p + \frac{(1-v_1^2)(1-v_3^2)}{4\pi^2} \int\int_{-\pi}^{\pi} X(\omega, \omega') s_n(\omega, \omega') \, \mathrm{d}\omega \, \mathrm{d}\omega', \tag{37}$$

where

$$X(\omega, \omega') = I_4(\omega) I_1(\omega') - I_3(\omega) I_2(\omega'), \tag{38}$$

$$s_n(\omega, \omega') = \boldsymbol{m}^{\mathrm{T}}(\omega) \boldsymbol{A}^{-1} \boldsymbol{m}(\omega'), \tag{39}$$

and $\boldsymbol{m}(\omega)$ is an $(n-1)$-dimensional vector, with elements

$$m_\alpha(\omega) = \mathrm{e}^{-\mathrm{i}\alpha\omega}, \qquad \alpha = 1, \ldots, n-1. \tag{40}$$

Now we look at the integral (29) for p, and replace the integrand $f(\omega)$ by its even part $\frac{1}{2}(f(\omega) + f(-\omega))$. Using (34), (36), $A(\omega)A(-\omega) = 1$, and the identity

$$(y_1^2 - y_2^2) D(\omega) D(-\omega) = (1-v_3^2)[1 - v_2(1-v_1^2)(D(\omega) + D(-\omega))], \tag{41}$$

we find that the integrand is the sum of three analytic parts: those terms containing a factor $D(\omega)$, those containing $D(-\omega)$, and those containing neither. Negating ω in the second, we obtain

$$p = -\frac{1}{2v_1} + \frac{1}{4\pi v_1(1-v_3^2)} \int_{-\pi}^{\pi} D(y_2 A - y_1)(A^{-1} + v_3) \, \mathrm{d}\omega. \tag{42}$$

From (38), (31), (34) and (35) we find that

$$X(\omega, \omega') = \frac{1}{2v_1(1-v_3^2)^2} \left(v_3 y_1 \frac{D'-D}{A'} + (v_3 y_2 D - y_1 D') \frac{A}{A'} + y_1 D - v_3 y_2 D' + y_2 (D'-D) A \right), \tag{43}$$

where we have written A, A' for $A(\omega)$, $A(\omega')$, and similarly for D.

The right-hand side of (43) separates into two analytic parts: one composed of all terms containing a factor D, the other a factor D'. Substituting into (37) and interchanging ω, ω' in the second part of X, we obtain

$$\begin{aligned} \mathscr{R}_n = v_1 - (1-v_1^2)p + \frac{1-v_1^2}{8\pi^2 v_1(1-v_3^2)} \int\int_{-\pi}^{\pi} & D(\omega)(y_2 A(\omega) - y_1)(v_3 - A(\omega')) \\ & \times (A(-\omega') s_n(\omega, \omega') - A(-\omega) s_n(\omega', \omega)) \, \mathrm{d}\omega \, \mathrm{d}\omega', \end{aligned} \tag{44}$$

where p is given by (42).

3. The limit $n \to \infty$

From (44) we see that $\mathscr{R}_n$ depends on n only via the function $s_n(\omega, \omega')$ defined by (39). Setting

$$z = e^{i\omega}, \qquad z' = e^{i\omega'}, \tag{45}$$

using the identity (16) and re-arranging, the definition (39) can be written as

$$s_n(\omega, \omega') = -|A|^{-1} \begin{vmatrix} a_0 & a_{-1} & \cdots & a_{-n+2} & z^{-1} \\ a_1 & a_0 & \cdots & a_{-n+1} & z^{-2} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-2} & a_{n-1} & \cdots & a_0 & z^{1-n} \\ z'^{-1} & z'^{-2} & \cdots & z'^{1-n} & 0 \end{vmatrix}. \tag{46}$$

The integrand in (44) is analytic in some strip surrounding the real axis in the ω and ω' planes. Thus there exists a positive ϵ such that we can replace the contour of integration by the line interval $(-i\epsilon - \pi, -i\epsilon + \pi)$. Both z and z' are then of modulus greater than one, and we expect $s_n(\omega, \omega')$ to tend to a limit as $n \to \infty$.

Grenander and Szego (1958, pp 40 and 51) have evaluated this limit for the case when the matrix A is Hermitian. Unfortunately this condition is not satisfied here, but the required generalization of their result appears to be the following.

Let A be any $(n-1) \times (n-1)$ Toeplitz matrix with elements $\alpha_{\beta-\alpha}$ such that the series

$$A(\omega) = \sum_{\alpha=-\infty}^{\infty} e^{i\alpha\omega} a_\alpha \tag{47}$$

is absolutely convergent in some strip containing the real axis in the ω plane, and $\ln A(\omega)$ is analytic and periodic of period 2π in this strip. Define $G(\omega)$, $H(\omega)$ such that

$$A(\omega) = G(\omega)H(\omega), \tag{48}$$

where $G(\omega)$ and $1/G(\omega)$ (($H(\omega)$ and $1/H(\omega)$)) are bounded, analytic and nonzero on the real axis and in the upper (lower) half-plane. (This is a Wiener–Hopf factorization—it is unique to within a multiplication constant.) Define $s_n(\omega, \omega')$ by (46). Then provided ω, ω', both have a negative imaginary part,

$$\lim_{n\to\infty} s_n(\omega, \omega') = [(zz'-1)H(\omega)G(-\omega')]^{-1}. \tag{49}$$

From (22), our function $A(\omega)$ can be factored into the form

$$A(\omega) = \left(\frac{(1-c_1 e^{i\omega})(1-c_2 e^{-i\omega})}{(1-c_1 e^{-i\omega})(1-c_2 e^{i\omega})}\right)^{1/2}. \tag{50}$$

For $T < T_c$, both c_1 and c_2 can be chosen to be positive and less than one. It follows that $A(\omega)$ satisfies the conditions of the above theorem, and by inspection we see that

$$G(\omega) = \left(\frac{1-c_1 e^{i\omega}}{1-c_2 e^{i\omega}}\right)^{1/2}, \tag{51}$$

$$H(\omega) = 1/G(-\omega). \tag{52}$$

Taking the limit $n \to \infty$, substituting the result (49) into (44), and using (48), (52), the integrand in (44) becomes

$$(zz'-1)^{-1}D(y_2A-y_1)\left[\frac{G'}{G}-\frac{H'}{H}+v_3\left(\frac{1}{HG'}-\frac{1}{GH'}\right)\right], \tag{53}$$

where $D = D(\omega)$, $G = G(\omega)$, $G' = G(\omega')$, etc.

We now use (45) to transform the ω' integration to an integration round a circle C in the z' plane. Due to our change of contour, this circle is slightly larger than the unit circle, and the pole of the integrand at $z' = z^{-1}$ lies inside C.

The terms containing H' in (53) are analytic on and outside C, and their contribution to the z' integrand tends to zero like z'^{-2} as $z' \to \infty$. They therefore give zero contribution to the integral.

The terms containing G' in (53) are analytic inside and on C, except for simple poles at $z' = z^{-1}$, 0. Their integrals can therefore be evaluated by the calculus of residues, giving (using (48) and (52))

$$\mathscr{R}_\infty = v_1-(1-v_1^2)p+\frac{1-v_1^2}{4\pi v_1(1-v_3^2)}\int_{-\pi}^{\pi} D(y_2A-y_1)(A^{-1}+v_3-G^{-1}-v_3H^{-1})\,\mathrm{d}\omega. \tag{54}$$

The integrand in (54) is analytic on and near the real axis, so we have shifted the contour of integration in the ω plane back to the real axis.

Using now the expression (42) for p, we find that the terms containing a factor $A^{-1}+v_3$ cancel. This is extremely fortunate, for these are the only terms we cannot integrate using elementary functions. We are left with (using (48))

$$\mathscr{R}_\infty = \tfrac{1}{2}(v_1+v_1^{-1})-\frac{1-v_1^2}{4\pi v_1(1-v_3^2)}\int_{-\pi}^{\pi} D(y_2H+v_3y_2G-y_1G^{-1}-v_3y_1H^{-1})\,\mathrm{d}\omega. \tag{55}$$

The terms containing H in (55) are meromorphic functions of z outside the unit circle (including the point at infinity), while those containing G are meromorphic within the unit circle. Thus each term can be integrated by using the calculus of residues, giving

$$\mathscr{R}_\infty = \tfrac{1}{2}(v_1+v_1^{-1}+v_2+v_2^{-1})+\frac{v_3y_2G_0-y_1G_0^{-1}}{2v_1v_2(1-v_3^2)}, \tag{56}$$

where G_0 is the value of $G(\omega)$ at the pole of $D(\omega)$. (If necessary, we deform the contour of integration in the z plane so as to enclose this pole.) Thus from (33) and (51),

$$G_0 = \left(\frac{v_1(1-v_2^2)+c_1v_2(1-v_1^2)}{v_1(1-v_2^2)+c_2v_2(1-v_1^2)}\right)^{1/2} \tag{57}$$

From (22) and (50), c_1 and c_2^{-1} are the roots of the quadratic equation

$$cx^2-ax+b = 0. \tag{58}$$

Hence, using (23),

$$c_1/c_2 = b/c = v_3^2. \tag{59}$$

Eliminating c_2 between (57) and (59), we can write c_1 as a bilinear function of G_0^2. Substituting this expression into (58), we obtain a quadratic equation for G_0^2. After some algebra, we find this equation can be written as

$$(y_2G_0-v_3y_1G_0^{-1})^2 = 4v_1^2v_2^2(1-v_3^2)^2. \tag{60}$$

By considering the limit $v_1 \to 1$ and using continuity arguments, we can verify that the correct solution of (60) is given by

$$y_2 G_0 - v_3 y_1 G_0^{-1} = 2v_1 v_2(1-v_3^2). \tag{61}$$

Taking the positive root of this equation for G_0 and substituting into (56), we finally obtain (using (32))

$$\begin{aligned}\mathscr{R}_\infty = {} & \tfrac{1}{2}(v_1+v_1^{-1}+v_2+v_2^{-1}+v_3+v_3^{-1}) \\ & -\tfrac{1}{2}(v_1v_2v_3)^{-1}[(1+v_1v_2v_3)(v_1+v_2v_3)(v_2+v_3v_1)(v_3+v_1v_2)]^{1/2}.\end{aligned} \tag{62}$$

Together with (9), this gives the 'three-spin magnetization' M_3 for the anisotropic triangular lattice. It is of course a symmetric function of v_1, v_2, v_3. The result (5) for the isotropic lattice follows by setting $v_1 = v_2 = v_3 = \tanh(J/kT)$.

4. Three-spin correlation on the square lattice

Another interesting case is the limit when $J_3 = v_3 = 0$. From figure 1, the diagonal interactions then disappear and M_3 becomes a three-spin correlation round a corner of the square lattice. From (9) and (62) we obtain

$$M_3 = M[1-4u_1u_2/(1-u_1)(1-u_2)], \tag{63}$$

where $u_i = \exp(-4\beta J_i)$ and M is the usual magnetization of the square lattice. Other three-spin correlations have been evaluated by Pink (1968).

Acknowledgments

The author thanks Drs J Stephenson and M F Sykes for helpful discussions regarding this problem.

References

Baxter R J 1972 *Ann. Phys., NY* **70** 193–228
Baxter R J, Sykes M F and Watts M G 1975 *J. Phys.* A: *Math. Gen.* **8** 245–51
Baxter R J and Wu F Y 1973 *Phys. Rev. Lett.* **31** 1294–7
—— 1974 *Aust. J. Phys.* **27** 357–81
Green H S 1963 *Z. Phys.* **171** 129–48
Grenander U and Szego G 1958 *Toeplitz Forms and Their Applications* (University of California Press)
Houtappel R M F 1950 *Physica* **16** 425–55
Husimi K and Syozi I 1950 *Prog. Theor. Phys.* **5** 177–186, 341–351
Kadanoff L P and Wegner F J 1971 *Phys. Rev.* B **4** 3989–93
Pink D A 1968 *Can. J. Phys.* **46** 2399–405
Potts R B 1955 *Proc. Phys. Soc.* A **68** 145–8
Stephenson J 1964 *J. Math. Phys.* **5** 1009–24
Sykes M F and Watts M G 1975 *J. Phys.* A: *Math. Gen.* **8** 1469–79
Wannier G H 1950 *Phys. Rev.* **79** 357–64
Wood D W and Griffiths H P 1972 *J. Phys.* C: *Solid St. Phys.* **5** L253–5
Wu F Y 1971 *Phys. Rev.* B **4** 2312–4